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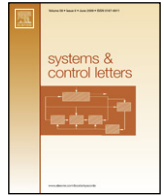
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A behavioral characterization of the positive real and bounded real characteristic values in balancing

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ABSTRACT

In passivity preserving and bounded realness preserving model reduction by balanced truncation, an important role is played by the so-called positive real (PR) and bounded real (BR) characteristic values. Both for the positive real as well as the bounded real case, these values are defined in terms of the extremal solutions of the algebraic Riccati associated with the system, more precisely as the square roots of the eigenvalues of the product matrix obtained by multiplying the smallest solution with the inverse of the largest solution of the Riccati equation. In this paper we will establish a *representation free* characterization of these values in terms of the behavior of the system. We will consider positive realness and bounded realness as special cases of half line dissipativity of the behavior. We will then show that both for the PR and the BR case, the characteristic values coincide with the singular values of the linear operator that assigns to each past trajectory in the input–output behavior its unique maximal supply extracting future continuation. We will explain that the term ‘singular values’ should be interpreted here in a generalized sense, since in our setup the future behavior is only an indefinite inner product space.

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1. Introduction

In classical Lyapunov balancing for input–output systems, an important role is played by the singular values of the Hankel operator, known as the *Hankel singular values*; see [1–3]. In terms of a state space representation of the system, these can be computed as the square roots of the eigenvalues of the product of the controllability Gramian and the observability Gramian. In the context of positive realness and bounded realness preserving model reduction by balancing, an equally important role is played by the so-called positive real (PR) characteristic values and bounded real (BR) characteristic values; see [4–9,2,10]. These are usually defined in terms of real symmetric solutions of certain linear matrix inequalities or algebraic Riccati equations associated with a state space representation of the system. More precisely, the PR and BR characteristic values are usually defined as the square roots of the eigenvalues of the product of the inverse maximal solution and the minimal solution of the algebraic Riccati equation. To the best of the author’s knowledge, no intrinsic input–output like characterization of the PR and BR characteristic values is known in the literature up until now.

In this paper, we will show that both the PR and BR characteristic values allow such an intrinsic, representation free

characterization in terms of the *behavior* and the *available storage* of the input–output system. The role of the Hankel operator of Lyapunov balancing is taken over in this context by the linear operator that assigns to each past input–output trajectory the unique future input–output trajectory that extracts the maximal amount of supply from the system. It will be shown that this operator has finite rank, and that its singular values coincide with the PR or BR characteristic values, of course depending on the choice of supply rate. It will be argued that the term ‘singular values’ should be interpreted here in a generalized sense, since the future behavior will only be an indefinite inner product space.

The approach in this paper will be to consider PR and BR input–output systems as special cases of systems that are *dissipative on the negative half line*, and whose number of input components is equal to the positive signature of the supply rate.

Results in the framework of behavioral balancing can also be found in the work of Weiland [11]. There, the classical problem of model reduction by balancing, without preservation of positive realness or bounded realness, was put into a more general behavioral framework for the first time. It was shown that the system invariants that appear as diagonal elements in the solutions of the algebraic Riccati equations after balancing are, in fact, the nonzero singular values of the operator from past to future behavior that assigns to each past trajectory its optimal continuation, with optimality in the sense of minimal weighted \mathcal{L}_2 norm of the future trajectory. Our work in the present paper can be regarded as an extension of Weiland’s work to the context of PR and BR preserving model reduction by balancing.

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At this point we also mention the recent paper [12] in which for a given PR or BR input–output system the PR and BR characteristic values were expressed in terms of the Hankel singular values of a particular, normalized, driving variable representation of the input–output system.

The outline of this paper is as follows. In Section 2 we review the basic material on dissipative linear differential systems, and on driving variable representations of behaviors. In Section 3 we introduce two linear operators associated with any strictly half line dissipative system. The first operator assigns to each past trajectory the unique future trajectory that extracts the available storage from the system, the second operator assigns to each future trajectory the unique past trajectory that yields the required supply. We formulate a theorem about singular value like decompositions of these operators, and introduce the notion of Σ -characteristic values. In Section 4 we use driving variable representations to prove this theorem, and to show that the Σ -characteristic values can be computed in terms of the extremal real symmetric solutions of the algebraic Riccati equation. Then, in the short Section 5, we apply our results to the special cases of positive real and bounded real input–output systems. The paper ends with some conclusions in Section 6.

Notation and background material. $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w)$ denotes the space of all infinitely often differentiable functions from \mathbb{R} to \mathbb{R}^w . $\mathcal{D}(\mathbb{R}, \mathbb{R}^w)$ denotes its subspace of functions with compact support. For this space we use the shorthand notation \mathcal{D} . We denote by $\mathcal{L}_2^{\text{loc}}(\mathbb{R}, \mathbb{R}^w)$ the space of all measurable functions w from \mathbb{R} to \mathbb{R}^w such that $\int_a^b \|w\|^2 dt < \infty$ for all $a, b \in \mathbb{R}$. $\mathcal{L}_2(\mathbb{R}, \mathbb{R}^w)$ denotes the ambient space of all measurable functions w from \mathbb{R} to \mathbb{R}^w such that $\int_{-\infty}^{\infty} \|w\|^2 dt < \infty$. The \mathcal{L}_2 -norm of w is $\|w\|_2 := (\int_{-\infty}^{\infty} \|w\|^2 dt)^{1/2}$. We denote by \mathbb{R}_- the set of negative real numbers, and by \mathbb{R}_+ the complementary set of nonnegative real numbers. $\mathcal{L}_2(\mathbb{R}_-, \mathbb{R}^w)$ ($\mathcal{L}_2(\mathbb{R}_+, \mathbb{R}^w)$) denotes the space of all measurable functions w from \mathbb{R}_- (\mathbb{R}_+) to \mathbb{R}^w such that $\int_{-\infty}^0 \|w\|^2 dt < \infty$ ($\int_0^{\infty} \|w\|^2 dt < \infty$). When the dimension of the co-domain is clear from the context, we denote these spaces by $\mathcal{L}_2(\mathbb{R})$, $\mathcal{L}_2(\mathbb{R}_-)$ and $\mathcal{L}_2(\mathbb{R}_+)$. For a given function w on \mathbb{R} , we denote by $w|_{\mathbb{R}_-}$ and $w|_{\mathbb{R}_+}$ the restrictions of w to \mathbb{R}_- and \mathbb{R}_+ , respectively. \mathbb{C}^- (\mathbb{C}^+) is the subset of \mathbb{C} of all λ such that $\text{Re}(\lambda) < 0$ ($\text{Re}(\lambda) > 0$). For a given square matrix $M \in \mathbb{R}^{w \times w}$, we denote by $\sigma(M)$ its set of eigenvalues. For a given nonsingular, symmetric matrix $\Sigma \in \mathbb{R}^{w \times w}$ we denote by $\sigma_+(\Sigma)$ (the *positive signature of Σ*) the number of positive eigenvalues of Σ .

2. Dissipative linear differential behaviors

In this paper we deal with *dissipative linear differential systems*. A subspace $\mathfrak{B} \subset \mathcal{L}_2^{\text{loc}}(\mathbb{R}, \mathbb{R}^w)$ is called a linear differential system (or a linear differential behavior) if it is equal to the space of (weak) solutions $w : \mathbb{R} \rightarrow \mathbb{R}^w$ of a system of linear, constant coefficient, higher order differential equations, i.e., there exists a polynomial matrix $R \in \mathbb{R}^{w \times w}[\xi]$ such that $\mathfrak{B} = \{w \in \mathcal{L}_2^{\text{loc}}(\mathbb{R}, \mathbb{R}^w) \mid R(\frac{d}{dt})w = 0\}$ (see [13]). The variable w is called the *manifest variable* of the system \mathfrak{B} . The set of all linear differential systems with w variables is denoted by \mathcal{L}^w .

For a given system \mathfrak{B} , a partition of the manifest variable w into $w = \text{col}(w_1, w_2)$ is called an *input–output partition* if w_1 is maximally free, meaning that it is free (i.e. for any $w_1 \in \mathcal{L}_2^{\text{loc}}(\mathbb{R}, \mathbb{R}^{w_1})$ there exists w_2 such that $\text{col}(w_1, w_2) \in \mathfrak{B}$), and one cannot enlarge the vector w_1 by adding one or more components of w_2 such that the enlarged variable is still free. If $w = \text{col}(w_1, w_2)$ is an input–output partition then w_1 is called input and w_2 is called output of \mathfrak{B} . For details we refer to [13]. The number of input components in any input–output partition of $\mathfrak{B} \in \mathcal{L}^w$ is an integer

invariant of \mathfrak{B} , and is called the *input cardinality* of \mathfrak{B} , denoted by $\mathfrak{m}(\mathfrak{B})$.

We restrict ourselves to controllable behaviors in this paper. A behavior $\mathfrak{B} \in \mathcal{L}^w$ is called *controllable* if for all $w_1, w_2 \in \mathfrak{B}$ there exists $T \geq 0$ and $w \in \mathfrak{B}$ such that $w(t) = w_1(t)$ for $t < 0$, and $w(t) = w_2(t - T)$ for $t \geq 0$. Properties of controllable behaviors are discussed in [13]. $\mathcal{L}_{\text{cont}}^w$ (a subset of \mathcal{L}^w) will denote the set of controllable behaviors.

Here, we will only review the basic material on dissipative behaviors. For an extensive treatment we refer to [14–17]. Let $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^w$ and let $\Sigma = \Sigma^\top \in \mathbb{R}^{w \times w}$ be nonsingular. The quadratic form $w^\top \Sigma w$ is called a *supply rate*.

Definition 1. $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^w$ is said to be Σ -dissipative if $\int_{-\infty}^{+\infty} w^\top \Sigma w dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathcal{D}$. \mathfrak{B} is said to be Σ -dissipative on \mathbb{R}_- if $\int_{-\infty}^0 Q_\Sigma(w) dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathcal{D}$. We will also call such behavior *half line dissipative*.

Using time-invariance, it is easily seen that if \mathfrak{B} is Σ -dissipative on \mathbb{R}_- , then it is Σ -dissipative.

Definition 2. $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^w$ is said to be *strictly Σ -dissipative* if there exists an $\epsilon > 0$ such that \mathfrak{B} is $(\Sigma - \epsilon I)$ -dissipative. It is said to be *strictly Σ -dissipative on \mathbb{R}_-* if there exists an $\epsilon > 0$ such that \mathfrak{B} is $(\Sigma - \epsilon I)$ -dissipative on \mathbb{R}_- .

If \mathfrak{B} is strictly Σ -dissipative on \mathbb{R}_- , then it is strictly Σ -dissipative.

In this paper we deal with linear differential behaviors $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^w$ that are strictly Σ -dissipative on \mathbb{R}_- . In addition, we will assume that $\mathfrak{m}(\mathfrak{B}) = \sigma_+(\Sigma)$, i.e. the input cardinality of \mathfrak{B} is equal to the positive signature of Σ . Two important special cases of such systems are

1. strictly positive real input–output systems, where the manifest variable is partitioned as $w = \text{col}(u, y)$, with u input and y output (having the same dimension \mathfrak{m}), and where $\Sigma = \begin{pmatrix} 0 & I_{\mathfrak{m}} \\ I_{\mathfrak{m}} & 0 \end{pmatrix}$, and
2. strictly bounded real input–output systems, where $w = \text{col}(u, y)$, with u input and y output, and where $\Sigma = \text{diag}(I_{\mathfrak{m}}, -I_{\mathfrak{p}})$.

It is well known that any $\mathfrak{B} \in \mathcal{L}^w$ admits state space representations. In this paper we will use mainly one type of state space representation, namely *driving variable representations* (DV-representations); see e.g. [18,19,11]. Consider the equations

$$\dot{x} = Ax + Bv, \quad w = Cx + Dv, \quad (1)$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{w \times n}$ and $D \in \mathbb{R}^{w \times m}$. These equations represent the *full behavior*

$$\mathfrak{B}_{DV}(A, B, C, D) := \{(w, x, v) \in \mathcal{L}_2^{\text{loc}}(\mathbb{R}, \mathbb{R}^w) \times \mathcal{L}_2^{\text{loc}}(\mathbb{R}, \mathbb{R}^n) \times \mathcal{L}_2^{\text{loc}}(\mathbb{R}, \mathbb{R}^m) \mid (1) \text{ holds}\}.$$

The variable x is a state variable, taking its values in \mathbb{R}^n , the state space, and v is called the *driving variable*, taking its values in \mathbb{R}^m . The *external behavior* corresponding to this full behavior is defined as

$$\mathfrak{B}_{DV}(A, B, C, D)_{\text{ext}} = \{w \in \mathcal{L}_2^{\text{loc}}(\mathbb{R}, \mathbb{R}^w) \mid \exists x \text{ and } v \text{ such that } (w, x, v) \in \mathfrak{B}_{DV}(A, B, C, D)\}.$$

If $\mathfrak{B} = \mathfrak{B}_{DV}(A, B, C, D)_{\text{ext}}$ then we call $\mathfrak{B}_{DV}(A, B, C, D)$ a *driving variable representation* of \mathfrak{B} .

An important integer invariant associated with a given behavior \mathfrak{B} is the minimal dimension of the state space over all its state space representations. This integer is called the *McMillan degree* of \mathfrak{B} , denoted with $\mathfrak{n}(\mathfrak{B})$.

A driving variable representation $\mathfrak{B}_{DV}(A, B, C, D)$ of \mathfrak{B} , with state space dimension n and driving variable dimension m is called *minimal* if n and m are minimal over all such driving variable

representations. The minimal n is equal to the McMillan degree $n(\mathfrak{B})$ and the minimal m is equal to the input cardinality $m(\mathfrak{B})$.

A given DV-representation $\mathfrak{B}_{DV}(A, B, C, D)$ of \mathfrak{B} is a minimal DV-representation if and only if (A, B, C, D) is strongly observable (meaning that the pair $(C + DF, A + BF)$ is observable for every F) and D has full column rank (see [18,19,11]). If $\mathfrak{B}_{DV}(A, B, C, D)$ is a minimal DV-representation of \mathfrak{B} , then \mathfrak{B} is controllable if and only if the pair (A, B) is controllable; see [11].

If we are dealing with an input–output system with input u and output y , represented in input–output state representation by

$$\dot{x} = Ax + Bu, \quad y = \bar{C}x + \bar{D}u, \quad (2)$$

then obviously a driving variable representation of the input–output behavior is given by

$$\dot{x} = Ax + Bv, \quad \begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{C} \end{pmatrix} x + \begin{pmatrix} I \\ \bar{D} \end{pmatrix} v.$$

3. Σ -characteristic values of system behaviors

In this section we introduce the notion of Σ -characteristic values of behaviors that are strictly Σ -dissipative on \mathbb{R}_- and that have the property $m(\mathfrak{B}) = \sigma_+(\Sigma)$, i.e., the input cardinality of \mathfrak{B} is equal to the positive signature of Σ .

We will use the property that \mathfrak{B} is strictly dissipative on the negative half line \mathbb{R}_- to endow the past behavior with an inner product, with the inner product given by the integral of the supply rate. In the same way, the supply rate will only yield an *indefinite* inner product on the future behavior. We will then formulate a theorem on singular value decompositions of two important operators between past and future behavior. The terminology ‘singular value’ should however be interpreted carefully, since the future behavior is not an inner product space. The ‘singular values’ will form a set of invariants of the strictly Σ -dissipative behavior, that will be called the Σ -characteristic values of \mathfrak{B} .

For any behavior $\mathfrak{B} \in \mathcal{L}^w$ we introduce the following notation:

$$\mathfrak{B}_- := \{w|_{\mathbb{R}_-} \mid w \in \mathfrak{B}\}, \quad \mathfrak{B}_+ := \{w|_{\mathbb{R}_+} \mid w \in \mathfrak{B}\}.$$

Furthermore, for a given past trajectory $w_- \in \mathfrak{B}_-$ define the set of all future trajectories w_+ whose concatenation at time zero with past trajectory w_- is in \mathfrak{B} by

$$\mathfrak{B}_+(w_-) := \{w_+ \in \mathfrak{B}_+ \mid \text{there exists } w \in \mathfrak{B} \text{ such that } w|_{\mathbb{R}_-} = w_- \text{ and } w|_{\mathbb{R}_+} = w_+\}.$$

For a given future trajectory $w_+ \in \mathfrak{B}_+$ define the set of all past trajectories w_- whose concatenation at time zero with future trajectory w_+ is in \mathfrak{B} by

$$\mathfrak{B}_-(w_+) := \{w_- \in \mathfrak{B}_- \mid \text{there exists } w \in \mathfrak{B} \text{ such that } w|_{\mathbb{R}_-} = w_- \text{ and } w|_{\mathbb{R}_+} = w_+\}.$$

Now, let $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^w$ and let a supply rate be given by the non-singular symmetric matrix $\Sigma = \Sigma^\top \in \mathbb{R}^{w \times w}$. Assume \mathfrak{B} is strictly Σ -dissipative. For a given past trajectory $w_- \in \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-)$ we define the associated *available storage* by

$$V_{av}(w_-) := \sup \left\{ - \int_0^\infty w_+^\top \Sigma w_+ dt \mid w_+ \in \mathfrak{B}_+(w_-) \cap \mathcal{L}_2(\mathbb{R}_+) \right\}, \quad (3)$$

and for a given future trajectory $w_+ \in \mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+)$ we define the associated *required supply* by

$$V_{req}(w_+) := \inf \left\{ \int_{-\infty}^0 w_-^\top \Sigma w_- dt \mid w_- \in \mathfrak{B}_-(w_+) \cap \mathcal{L}_2(\mathbb{R}_-) \right\}. \quad (4)$$

The available storage associated with past trajectory w_- is the maximal amount of supply that can be extracted from the system over all future trajectories $w_+ \in \mathfrak{B}_+(w_-) \cap \mathcal{L}_2(\mathbb{R}_+)$. The required supply associated with future trajectory w_+ is the minimal amount of supply that has to be delivered to the system over all past trajectories $w_- \in \mathfrak{B}_-(w_+) \cap \mathcal{L}_2(\mathbb{R}_-)$.

Due to Σ -dissipativity of \mathfrak{B} , the supremum and infimum above are finite for all w_- and w_+ , respectively (see [14,16,17]). Also, by *strict* Σ -dissipativity, both the supremum and infimum are attained for all w_- and w_+ . In particular, for given $w_- \in \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-)$ there is a *unique* $w_+^* \in \mathfrak{B}_+(w_-) \cap \mathcal{L}_2(\mathbb{R}_+)$ such that

$$V_{av}(w_-) = - \int_0^\infty w_+^{*\top} \Sigma w_+^* dt$$

and for given $w_+ \in \mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+)$ there is a *unique* $w_-^* \in \mathfrak{B}_-(w_+) \cap \mathcal{L}_2(\mathbb{R}_-)$ such that

$$V_{req}(w_+) = \int_{-\infty}^0 w_-^{*\top} \Sigma w_-^* dt.$$

By associating with any past trajectory $w_- \in \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-)$ the unique optimal future trajectory $w_+^* \in \mathfrak{B}_+(w_-) \cap \mathcal{L}_2(\mathbb{R}_+)$ we obtain the operator

$$\Gamma_- : \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-) \rightarrow \mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+), \quad \Gamma_-(w_-) = w_+^*,$$

and by associating with any future trajectory $w_+ \in \mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+)$ the unique optimal past trajectory $w_-^* \in \mathfrak{B}_-(w_+) \cap \mathcal{L}_2(\mathbb{R}_-)$ we obtain the operator

$$\Gamma_+ : \mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+) \rightarrow \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-), \quad \Gamma_+(w_+) = w_-^*.$$

In the above, we have only assumed that our behavior \mathfrak{B} is strictly Σ -dissipative. In the remainder of this section we will now make the *stronger assumption* that \mathfrak{B} is strictly Σ -dissipative on \mathbb{R}_- . It can be shown that this is equivalent with the existence of $\epsilon > 0$ such that

$$\int_{-\infty}^0 w^\top \Sigma w dt \geq \epsilon \int_{-\infty}^0 \|w\|^2 dt$$

for all $w \in \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-)$. This immediately implies that the bilinear form

$$\langle w_1, w_2 \rangle_{-, \Sigma} := \int_{-\infty}^0 w_1^\top \Sigma w_2 dt$$

defines an inner product on $\mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-)$. On $\mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+)$ we define the bilinear form

$$\langle w_1, w_2 \rangle_{+, \Sigma} := - \int_0^\infty w_1^\top \Sigma w_2 dt.$$

Since there are no assumptions on the average supply over the future behavior, this only defines an *indefinite* inner product on $\mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+)$.

Now, in what follows it will be shown that the operators Γ_- and Γ_+ are linear. We will denote by $\Gamma_-^* : \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-) \rightarrow \mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+)$ the *adjoint* of Γ_- , i.e. the (unique) linear operator $\Gamma_-^* : \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-) \rightarrow \mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+)$ that satisfies

$$\langle w_1, \Gamma_-(w_2) \rangle_{+, \Sigma} = \langle \Gamma_-^*(w_1), w_2 \rangle_{-, \Sigma}$$

for all $w_1 \in \mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+)$ and $w_2 \in \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-)$. The existence and uniqueness of this adjoint (in an indefinite inner product context) can be easily proven; see e.g. [20], chapter 4. Likewise, $\Gamma_+^* : \mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+) \rightarrow \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-)$ will denote the adjoint of Γ_+ , i.e. the unique linear operator that satisfies

$$\langle w_1, \Gamma_+(w_2) \rangle_{-, \Sigma} = \langle \Gamma_+^*(w_1), w_2 \rangle_{+, \Sigma}$$

for all $w_1 \in \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-)$ and $w_2 \in \mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+)$.

We now formulate a theorem stating that if \mathfrak{B} is strictly Σ -dissipative on \mathbb{R}_- and, in addition, $\mathfrak{m}(\mathfrak{B}) = \sigma_+(\Sigma)$, then the operators Γ_- and Γ_+ allow *singular value decompositions* that, in a certain sense, are compatible. It should however be understood that, strictly speaking, the terminology singular value decomposition is not appropriate in the present context, since our operators do not act between genuine inner product spaces: only the past behavior is an inner product space, on the future behavior we have an indefinite inner product. The notion singular value should therefore be interpreted in a generalized sense:

Theorem 3. Assume that $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^w$ is strictly Σ -dissipative on \mathbb{R}_- and $\mathfrak{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. The operators Γ_- and Γ_+ are linear. The operator $\Gamma_-^* \Gamma_- : \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-) \rightarrow \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-)$ has a finite-dimensional image, and it is Hermitian and nonnegative. There exist positive real numbers $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$, where $n = \mathfrak{n}(\mathfrak{B})$, the McMillan degree of \mathfrak{B} , such that $\sigma_1^2 \geq \sigma_2^2 \geq \dots \geq \sigma_n^2 > 0$ are the nonzero eigenvalues of $\Gamma_-^* \Gamma_-$. There exists an orthonormal set $\{w_1^-, w_2^-, \dots, w_n^-\} \subset \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-)$, and an orthonormal set $\{w_1^+, w_2^+, \dots, w_n^+\} \subset \mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+)$ such that

$$\Gamma_- = \sum_{i=1}^n \sigma_i \langle \cdot, w_i^- \rangle_{-, \Sigma} w_i^+, \quad (5)$$

$$\Gamma_+ = \sum_{i=1}^n \frac{1}{\sigma_i} \langle \cdot, w_i^+ \rangle_{+, \Sigma} w_i^-. \quad (6)$$

A proof of this theorem will be given in Section 4 of this paper.

An important result in the above theorem is the nonnegativity of the map $\Gamma_-^* \Gamma_-$. Of course, in genuine inner product spaces this nonnegativity is trivially satisfied. In the present context, however, it is a statement that needs to be proven explicitly, and which will follow from the fact that the image of the operator Γ_- is a positive subspace for the indefinite future inner product. This follows from the nonnegativity of the available storage (which in turn follows from dissipativity on the negative half line and the assumption that $\mathfrak{m}(\mathfrak{B}) = \sigma_+(\Sigma)$).

Definition 4. The positive real numbers $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$ will be called the Σ -characteristic values of \mathfrak{B} .

As noted before, in a generalized sense these numbers are the singular values of the map Γ_- . In that sense, the pairs of functions (w_i^-, w_i^+) can be considered as Schmidt pairs of Γ_- .

In Section 5 we will prove that for the special cases of strict positive realness and strict bounded realness, the Σ -characteristic values defined here coincide with the PR characteristic values and BR characteristic values, respectively. This will follow immediately from the characterization of the Σ -characteristic values in terms of solutions of the algebraic Riccati equation in Section 4. The conclusion is that Theorem 3 applied to these special cases yields a behavioral, representation free characterization of the PR and BR characteristic values appearing in the literature on positive realness and bounded realness preserving balanced truncation.

Remark 5. In [11], an analogous theorem was proven in a slightly different context in which both past as well as future behavior were assumed to be inner product spaces. Using this genuine inner product structure, in [11] elementary least squares arguments were used to prove the theorem. In the present context, the proof given in [11] breaks down.

4. State space characterizations and representations

In this section we review the characterizations of (strict) Σ -dissipativity in terms of the algebraic Riccati equation associated

with a minimal DV-representation of the given behavior \mathfrak{B} . We explicitly compute representations of the linear operators (and their adjoints) that assign to each past (future) trajectory the unique state at time zero, and we characterize the extremal solutions of the Riccati equation in terms of these operators. We also compute the operators Γ_- and Γ_+ in terms of compositions of these operators. It will turn out that the Σ -characteristic values as defined in Definition 4 are the eigenvalues of the product of the inverse of the maximal solution and the minimal solution of the algebraic Riccati equation. Much of the material in this section is an extension of results in [11] to the case that the future behavior is an indefinite inner product space.

Proposition 6. Let $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^w$ with minimal DV-representation $\mathfrak{B}_{\text{DV}}(A, B, C, D)$ and let $\Sigma = \Sigma^\top \in \mathbb{R}^{w \times w}$ be nonsingular. Assume $D^\top \Sigma D > 0$. Then

1. \mathfrak{B} is Σ -dissipative if and only if there exists a real symmetric solution $P = P^\top \in \mathbb{R}^{n \times n}$ of the algebraic Riccati equation (ARE)
$$A^\top P + PA - C^\top \Sigma C + (PB - C^\top \Sigma D)(D^\top \Sigma D)^{-1} \times (B^\top P - D^\top \Sigma C) = 0. \quad (7)$$

If this is the case, then there exist real symmetric solutions P_- and P_+ such that every real symmetric solution P satisfies $P_- \leq P \leq P_+$.

2. \mathfrak{B} is Σ -dissipative on \mathbb{R}_- if and only if there exists a positive semidefinite solution $P = P^\top \in \mathbb{R}^{n \times n}$ of the ARE (7).
3. If $\mathfrak{m}(\mathfrak{B}) = \sigma_+(\Sigma)$ then \mathfrak{B} is Σ -dissipative on \mathbb{R}_- if and only if all solutions of ARE (7) are positive definite, equivalently $P_- > 0$.

Proof. (1) is proved in [14, Theorem 8.4.5] and (2), (3) are proved in [16, Theorem 6.4]. \square

Proposition 7. Let $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^w$ with minimal DV-representation $\mathfrak{B}_{\text{DV}}(A, B, C, D)$ and let $\Sigma = \Sigma^\top \in \mathbb{R}^{w \times w}$ be nonsingular. If \mathfrak{B} is strictly Σ -dissipative then $D^\top \Sigma D > 0$, and the minimal and maximal real symmetric solution P_- and P_+ of the ARE (7) satisfy $P_+ > P_-$. Furthermore, P_- and P_+ are stabilizing and anti-stabilizing, respectively, i.e., $\sigma(A_-) \subset \mathbb{C}^-$ and $\sigma(A_+) \subset \mathbb{C}^+$, where we denote

$$A_+ := A + B(D^\top \Sigma D)^{-1}(B^\top P_+ - D^\top \Sigma C), \quad (8)$$

$$A_- := A + B(D^\top \Sigma D)^{-1}(B^\top P_- - D^\top \Sigma C). \quad (9)$$

Finally, the following statements are equivalent:

1. \mathfrak{B} is strictly Σ -dissipative on \mathbb{R}_- ,
2. $D^\top \Sigma D > 0$ and the maximal solution P_+ of the ARE (7) is positive definite and anti-stabilizing, i.e., $\sigma(A_+) \subset \mathbb{C}^+$.

Proof. A proof of the claim that $P_+ > P_-$ is contained in the proof of Theorem 5.7 in [16]. Proofs of the statement $D^\top \Sigma D > 0$, and the equivalence of statements 1 and 2 can be given similar to the proof of Theorem 5.3.4 in [21]. There, it was also shown that strict Σ -dissipativity implies that the Hamiltonian matrix associated with the ARE has no imaginary eigenvalues. This implies that P_- and P_+ must be stabilizing and anti-stabilizing, respectively. We omit the details. \square

We will now study the maps Γ_- and Γ_+ in terms of DV-representations of the given behavior \mathfrak{B} . Let $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^w$ with minimal DV-representation $\mathfrak{B}_{\text{DV}}(A, B, C, D)$. Let $n = \mathfrak{n}(\mathfrak{B})$ be the McMillan degree of \mathfrak{B} . By minimality, for every $w \in \mathfrak{B}$ there is a unique state trajectory x . For any given $x_0 \in \mathbb{R}^n$, let $\mathfrak{B}(x_0)$ denote the set of all $w \in \mathfrak{B}$ such that the corresponding state trajectory x satisfies $x(0) = x_0$. Thus, for every $w \in \mathfrak{B}$ there is a unique $x_0 \in \mathbb{R}^n$ such that $w \in \mathfrak{B}(x_0)$. Moreover (see [11]), there

exists linear surjective maps $R_- : \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-) \rightarrow \mathbb{R}^n$ and $R_+ : \mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+) \rightarrow \mathbb{R}^n$ such that for all $x_0 \in \mathbb{R}^n$ we have

$$w \in \mathfrak{B}(x_0) \Leftrightarrow \{R_-(w_-) = x_0 \text{ and } R_+(w_+) = x_0\},$$

where $w_- := w|_{\mathbb{R}_-}$ and $w_+ := w|_{\mathbb{R}_+}$. In what follows we will explicitly compute representations of the maps R_- and R_+ , and their adjoints R_-^* and R_+^* in terms of the systems matrices A, B, C and D . On \mathbb{R}^n we take the standard Euclidean inner product. Note that R_+^* denotes the generalized adjoint with respect to the indefinite inner product on $\mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+)$.

It is well known (see [15]) that the extremal solutions of the Riccati equation (7) are associated with the available storage and required supply as reviewed in the previous section:

Proposition 8. Let $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^w$ with minimal DV-representation $\mathfrak{B}_{DV}(A, B, C, D)$. Assume that $D^\top \Sigma D > 0$. Assume \mathfrak{B} is Σ -dissipative and let P_- and P_+ be the minimal and maximal real symmetric solutions of the ARE (7). Then for any $w_- \in \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-)$ we have $V_{av}(w_-) = x_0^\top P_- x_0$, where $x_0 := R_-(w_-)$. Also, for any $w_+ \in \mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+)$ we have $V_{req}(w_+) = x_0^\top P_+ x_0$, where $x_0 := R_+(w_+)$.

If \mathfrak{B} is strictly Σ -dissipative then P_- and P_+ satisfy $\sigma(A_+) \subset \mathbb{C}^+$ and $\sigma(A_-) \subset \mathbb{C}^-$ (see Proposition 7). Introduce the following notation:

$$C_+ = C + D(D^\top \Sigma D)^{-1}(B^\top P_+ - D^\top \Sigma C), \quad (10)$$

$$C_- = C + D(D^\top \Sigma D)^{-1}(B^\top P_- - D^\top \Sigma C). \quad (11)$$

The following is also well known (see also [22]).

Proposition 9. Let $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^w$ with minimal DV-representation $\mathfrak{B}_{DV}(A, B, C, D)$. Assume \mathfrak{B} is strictly Σ -dissipative. Then for $w_- \in \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-)$ the unique optimal future trajectory w_+^* is given by $w_+^*(t) = C_- e^{A_- t} x_0$, where $x_0 := R_-(w_-)$. Also, for $w_+ \in \mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+)$ the unique optimal past trajectory w_-^* is given by $w_-^*(t) = C_+ e^{A_+ t} x_0$, where $x_0 := R_+(w_+)$.

In the remainder of this section we will assume that \mathfrak{B} is strictly Σ -dissipative on \mathbb{R}_- and that $\mathfrak{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. In that case, in addition we have $0 < P_- < P_+$ (see Propositions 6 and 7). The next theorem is the main result of this section. It computes representations of R_- and R_+ and their adjoints R_-^* and R_+^* , and shows that P_- and P_+ can be expressed in terms of compositions of these maps.

Theorem 10. Let $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^w$ with minimal DV-representation $\mathfrak{B}_{DV}(A, B, C, D)$. Assume that \mathfrak{B} is strictly Σ -dissipative on \mathbb{R}_- and that $\mathfrak{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Then for any $w_- \in \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-)$ and $w_+ \in \mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+)$ we have

$$R_-(w_-) = \int_{-\infty}^0 e^{-(A_+ - P_+^{-1} C_+^\top \Sigma C_+) \tau} P_+^{-1} C_+^\top \Sigma w_-(\tau) d\tau, \quad (12)$$

$$R_+(w_+) = - \int_0^\infty e^{-(A_- - P_-^{-1} C_-^\top \Sigma C_-) \tau} P_-^{-1} C_-^\top \Sigma w_+(\tau) d\tau. \quad (13)$$

Furthermore, for any $x_0 \in \mathbb{R}^n$ we have

$$R_-^*(x_0) = C_+ P_+^{-1} e^{-(A_+ - P_+^{-1} C_+^\top \Sigma C_+)^\top t} x_0 \quad (14)$$

and

$$R_+^*(x_0) = C_- P_-^{-1} e^{-(A_- - P_-^{-1} C_-^\top \Sigma C_-)^\top t} x_0. \quad (15)$$

Finally, $P_+ = (R_- R_-^*)^{-1}$ and $P_- = (R_+ R_+^*)^{-1}$.

Proof. It is easily verified from the ARE (7) that $A_+^\top P_+ + P_+ A_+ - C_+^\top \Sigma C_+ = 0$ and $A_-^\top P_- + P_- A_- - C_-^\top \Sigma C_- = 0$. This yields that P_+^{-1}, P_-^{-1} are the unique solutions to

$$P_+^{-1} A_+^\top + A_+ P_+^{-1} - P_+^{-1} C_+^\top \Sigma C_+ P_+^{-1} = 0 \quad (16)$$

and

$$P_-^{-1} A_-^\top + A_- P_-^{-1} - P_-^{-1} C_-^\top \Sigma C_- P_-^{-1} = 0, \quad (17)$$

respectively. We claim that $A_+ - P_+^{-1} C_+^\top \Sigma C_+$ is similar to $-A_+^\top$. Indeed, from (16) we have $-P_+^{-1} A_+^\top = (A_+ - P_+^{-1} C_+^\top \Sigma C_+) P_+^{-1}$ so

$$P_+(A_+ - P_+^{-1} C_+^\top \Sigma C_+) P_+^{-1} = -A_+^\top. \quad (18)$$

As a consequence, $\sigma(A_+ - P_+^{-1} C_+^\top \Sigma C_+) \subset \mathbb{C}^-$. In the same way we show that $\sigma(A_- - P_-^{-1} C_-^\top \Sigma C_-) \subset \mathbb{C}^+$. Also, note that $C_+^\top \Sigma D = P_+ B$ and $C_-^\top \Sigma D = P_- B$. It is easily seen that $\mathfrak{B}_{DV}(A_+, B, C_+, D)$ and $\mathfrak{B}_{DV}(A_-, B, C_-, D)$ both provide a minimal driving variable representation of \mathfrak{B} . We will now prove (12). Let $w_- \in \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-)$. There exist x, v such that $\dot{x} = A_+ x + Bv$ and $w_- = C_+ x + Dv$. This yields $C_+^\top \Sigma w_- = C_+^\top \Sigma C_+ x + C_+^\top \Sigma Dv = C_+^\top \Sigma C_+ x + P_+ Bv$. Consequently,

$$Bv = P_+^{-1} C_+^\top \Sigma w_- - P_+^{-1} C_+^\top \Sigma C_+ x.$$

Thus the state trajectory x corresponding to w_- satisfies $\dot{x} = (A_+ - P_+^{-1} C_+^\top \Sigma C_+)x + P_+^{-1} C_+^\top \Sigma w_-$. Since $A_+ - P_+^{-1} C_+^\top \Sigma C_+$ is stable, this implies that $x(0) = R_-(w_-)$ is given by (12). In the same way, working with the driving variable representation $\mathfrak{B}_{DV}(A_-, B, C_-, D)$, we can prove (13). We will now prove (14). Let $x_0 \in \mathbb{R}^n$ and let $w_- \in \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-)$. Let (x, y) denote the Euclidean inner product on \mathbb{R}^n . We have

$$(x_0, R_-(w_-)) = \int_{-\infty}^0 x_0^\top e^{-(A_+ - P_+^{-1} C_+^\top \Sigma C_+) \tau} P_+^{-1} C_+^\top \Sigma w_-(\tau) d\tau.$$

The latter should be equal to $\langle R_-^*(x_0), w_- \rangle_{-, \Sigma}$ so $R_-^*(x_0)$ must be given by (14). In the same way we can prove (15). Finally, for any $x_0 \in \mathbb{R}^n$ we have

$$(R_- R_-^*)(x_0) = \int_{-\infty}^0 e^{-(A_+ - P_+^{-1} C_+^\top \Sigma C_+) \tau} P_+^{-1} C_+^\top \Sigma C_+ P_+^{-1} \times e^{-(A_+ - P_+^{-1} C_+^\top \Sigma C_+)^\top \tau} d\tau x_0.$$

It is easily verified that the integral on the right is equal to the unique solution X of the equation

$$(A_+ - P_+^{-1} C_+^\top \Sigma C_+)X + X(A_+ - P_+^{-1} C_+^\top \Sigma C_+)^\top + P_+^{-1} C_+^\top \Sigma C_+ P_+^{-1} = 0,$$

which yields $X = P_+^{-1}$ by virtue of Eq. (16). We conclude that $R_- R_-^* = P_+^{-1}$. In the same way we can prove that $R_+ R_+^* = P_-^{-1}$. \square

Remark 11. In the case that both the past and the future behavior are inner product spaces a result analogous to $P_+ = (R_- R_-^*)^{-1}$ and $P_- = (R_+ R_+^*)^{-1}$ was proven in [11] using a general least squares argument, without computing explicit representations of R_- , R_-^* , R_+ and R_+^* .

Corollary 12. Let $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^w$ with minimal DV-representation $\mathfrak{B}_{DV}(A, B, C, D)$. Assume that \mathfrak{B} is strictly Σ -dissipative on \mathbb{R}_- and that $\mathfrak{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Then we have $\Gamma_- = R_+^*(R_+ R_+^*)^{-1} R_-$ and $\Gamma_+ = R_-^*(R_- R_-^*)^{-1} R_+$.

Proof. By Proposition 9 we know that for $w_- \in \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-)$ we have $\Gamma_-(w_-) = C_- e^{A_- t} x_0$, with $x_0 = R_-(w_-)$. By (17) we have $A_- = -P_-^{-1}(A_- - P_-^{-1} C_-^\top \Sigma C_-)^\top P_-$ so

$$\begin{aligned} \Gamma_-(w_-) &= C_- e^{-P_-^{-1}(A_- - P_-^{-1} C_-^\top \Sigma C_-)^\top t} x_0 \\ &= C_- P_-^{-1} e^{-(A_- - P_-^{-1} C_-^\top \Sigma C_-)^\top t} P_- x_0. \end{aligned}$$

By Theorem 10 this is equal to $(R_+^*(R_+ R_+^*)^{-1} R_-)(w_-)$. In the same way a proof for Γ_+ can be given. \square

We are now in a position to prove [Theorem 3](#).

Proof of Theorem 3. The claim that Γ_- and Γ_+ are linear follows immediately from [Corollary 12](#). Next, note that $\Gamma_-^* \Gamma_- = R_-^* (R_+ R_-^*)^{-1} R_-$. Since R_- is surjective and $R_+ R_-^1$ maps \mathbb{R}^n onto itself, $\Gamma_-^* \Gamma_-$ has an n -dimensional image and has therefore n nonzero eigenvalues (see [23]). It is easily verified that $\Gamma_-^* \Gamma_-$ is Hermitian. The fact that it is nonnegative uses the fact that the available storage is nonnegative: for any $w_- \in \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-)$ we have $\langle w_-, (\Gamma_-^* \Gamma_-)(w_-) \rangle_{-, \Sigma} = \langle \Gamma_-(w_-), \Gamma_-(w_-) \rangle_{+, \Sigma} = V_{av}(w_-) = x_0^T P_- x_0 \geq 0$, where $x_0 = R_-(w_-)$. From this it follows that $\Gamma_-^* \Gamma_-$ has n positive eigenvalues, say $\sigma_1^2 \geq \sigma_2^2 \geq \dots \geq \sigma_n^2$. By [23], Theorem 8.15 there exists an orthonormal set $\{w_1^-, w_2^-, \dots, w_n^-\}$ of eigenvectors, $(\Gamma_-^* \Gamma_-)(w_i^-) = \sigma_i^2 w_i^-$. Now define $w_i^+ \in \mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+)$ by $w_i^+ := \frac{1}{\sigma_i} \Gamma_-(w_i^-)$. We prove that $\{w_1^+, w_2^+, \dots, w_n^+\}$ is an orthonormal subset of $\mathfrak{B}_+ \cap \mathcal{L}_2(\mathbb{R}_+)$ (in the indefinite inner product). Indeed, $\langle w_i^+, w_j^+ \rangle_{+, \Sigma} = \langle \frac{1}{\sigma_i} \Gamma_-(w_i^-), \frac{1}{\sigma_j} \Gamma_-(w_j^-) \rangle_{+, \Sigma} = \langle \frac{1}{\sigma_i} w_i^-, \frac{1}{\sigma_j} (\Gamma_-^* \Gamma_-)(w_j^-) \rangle_{-, \Sigma} = \langle \frac{1}{\sigma_i} w_i^-, \sigma_j w_j^- \rangle_{-, \Sigma} = \delta_{ij}$, with δ_{ij} the Kronecker delta. From the orthonormality of the w_i^+ it also follows that they are linearly independent. By definition we have $\Gamma_-(w_i^-) = \sigma_i w_i^+$.

We now prove (5). Since the image of Γ_- is n -dimensional, the set $\{w_1^+, w_2^+, \dots, w_n^+\}$ (being linearly independent) forms a basis of this image. Let $w_- \in \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-)$. Then there exist μ_i such that $\Gamma_-(w_-) = \sum_{i=1}^n \mu_i w_i^+$. We compute the μ_i as follows: $\mu_i = \langle \Gamma_-(w_-), w_i^+ \rangle_{+, \Sigma} = \langle \Gamma_-(w_-), \frac{1}{\sigma_i} \Gamma_-(w_i^-) \rangle_{+, \Sigma} = \langle w_-, \frac{1}{\sigma_i} (\Gamma_-^* \Gamma_-)(w_i^-) \rangle_{-, \Sigma} = \langle w_-, \frac{1}{\sigma_i} \sigma_i^2 w_i^- \rangle_{-, \Sigma} = \sigma_i \langle w_-, w_i^- \rangle_{-, \Sigma}$. This proves (5).

Next, we prove (6). We first show that $\Gamma_+(w_i^+) = \frac{1}{\sigma_i} w_i^-$. By definition, $w_i^+ = \frac{1}{\sigma_i} \Gamma_-(w_i^-)$, so we have $\Gamma_+(w_i^+) = \frac{1}{\sigma_i} (\Gamma_+ \Gamma_-)(w_i^-)$. Also, by [Corollary 12](#), $\Gamma_+ \Gamma_- = R_-^* (R_+ R_-^*)^{-1} R_-$. Since w_i^- is an eigenvector of $\Gamma_-^* \Gamma_-$, we have $w_i^- \in \text{im}(\Gamma_-^*) \subset \text{im}(R_-^*)$. Hence there exists v_i such that $w_i^- = R_-^* v_i$. This implies that $\Gamma_+(w_i^+) = \frac{1}{\sigma_i} R_-^* (v_i) = \frac{1}{\sigma_i} w_i^-$. Finally, since Γ_+ has an n -dimensional image with basis $\{w_1^-, w_2^-, \dots, w_n^-\}$, the remainder of the proof can be given along the lines of the corresponding result for Γ_- . \square

We will now prove the key theorem of this paper, stating that the Σ -characteristic values of \mathfrak{B} , i.e., the eigenvalues $\sigma_1^2 \geq \sigma_2^2 \geq \dots \geq \sigma_n^2 > 0$ of $\Gamma_-^* \Gamma_-$, are in fact the eigenvalues of $P_+^{-1} P_-$, with $0 < P_- < P_+$ the extremal solutions of the ARE (7), for any minimal DV-representation of \mathfrak{B} :

Theorem 13. Assume that \mathfrak{B} is strictly Σ -dissipative on \mathbb{R}_- and $\mathfrak{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Let $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$ be the Σ -characteristic values of \mathfrak{B} . Let $\mathfrak{B}_{DV}(A, B, C, D)$ be a minimal DV-representation of \mathfrak{B} with $0 < P_- < P_+$ the extremal solutions of the ARE (7). Then $\{\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2\} = \sigma(P_+^{-1} P_-)$. Furthermore $0 < \sigma_i < 1$ for all i .

Proof. Let $\mathfrak{B}_{DV}(A, B, C, D)$ be a minimal driving variable representation of \mathfrak{B} . For $i = 1, 2, \dots, n$ there exist $w_i^- \in \mathfrak{B}_- \cap \mathcal{L}_2(\mathbb{R}_-)$ such that $(\Gamma_-^* \Gamma_-)(w_i^-) = \sigma_i^2 w_i^-$. This is equivalent with $(R_- R_-^* (R_+ R_-^*)^{-1} R_-)(w_i^-) = \sigma_i^2 w_i^-$. Now, $R_-(w_i^-) \neq 0$, for otherwise, by [Corollary 12](#), we would have $\Gamma_-(w_i^-) = 0$ implying $(\Gamma_-^* \Gamma_-)(w_i^-) = 0$ so $\sigma_i = 0$. Thus $\sigma_i^2 \in \sigma(P_+^{-1} P_-)$. Conversely, let λ and $x \neq 0$ be such that $P_+^{-1} P_- x = \lambda x$. Then $\Gamma_-^* \Gamma_- R_-^* (R_+ R_-^*)^{-1} x = \lambda R_-^* (R_+ R_-^*)^{-1} x$. By surjectivity of R_- we have $R_-^* (R_+ R_-^*)^{-1} x \neq 0$. Thus $\lambda = \sigma_i^2$ or some i . We finally prove $\sigma_i < 1$ for all i . From $0 < P_- < P_+$, we obtain $P_+^{-1} P_- P_+^{-\frac{1}{2}} < I$. The

claim follows from the fact that the eigenvalues of $P_+^{-\frac{1}{2}} P_- P_+^{-\frac{1}{2}}$ and $P_+^{-1} P_-$ coincide since the two matrices are similar. \square

To conclude this section, we deal with the question whether the results up until now can be extended to the case that our system, instead of strictly Σ -dissipative on \mathbb{R}_- , is only Σ -dissipative on \mathbb{R}_- . Clearly, strict Σ -dissipativity on \mathbb{R}_- implies Σ -dissipativity on \mathbb{R}_- . The following proposition makes the relation between these two properties explicit for the special case $\mathfrak{m}(\mathfrak{B}) = \sigma_+(\Sigma)$:

Proposition 14. Let $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^w$ with minimal DV-representation $\mathfrak{B}_{DV}(A, B, C, D)$ and let $\Sigma = \Sigma^T \in \mathbb{R}^{w \times w}$ be nonsingular. Assume that $\mathfrak{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Then the following statements are equivalent.

1. \mathfrak{B} is strictly Σ -dissipative on \mathbb{R}_- ,
2. \mathfrak{B} is Σ -dissipative on \mathbb{R}_- , $D^T \Sigma D > 0$ and the Hamiltonian matrix

$$H := \begin{pmatrix} A_1 & B_1 B_1^T \\ C_1^T \Sigma C_1 & -A_1^T \end{pmatrix}, \quad (19)$$

where $A_1 := A - B(D^T \Sigma D)^{-1} D^T \Sigma C$, $B_1 := B(D^T \Sigma D)^{-\frac{1}{2}}$, $C_1 := C - D(D^T \Sigma D)^{-1} D^T \Sigma C$, has no eigenvalues on the imaginary axis $\mathbb{C}^0 := \{\lambda \in \mathbb{C} \mid \text{Re}(\lambda) = 0\}$

Proof. First note that the set of eigenvalues $\sigma(H)$ of the Hamiltonian H is the union of $\sigma(A_+)$, with A_+ given by (8), and its mirror image in the imaginary axis. The implication (1) \Rightarrow (2) then follows from the implication (1) \Rightarrow (2) in [Proposition 7](#). We now prove the converse, (2) \Rightarrow (1). Since $\mathfrak{m}(\mathfrak{B}) = \sigma_+(\Sigma)$, by [Proposition 6](#), statement 3, we have $P_+(> P_-) > 0$. Since, in addition, P_+ is anti-stabilizing and $D^T \Sigma D > 0$, [Proposition 7](#) implies that \mathfrak{B} is strictly Σ -dissipative on \mathbb{R}_- . Maximal solution P_+ of the ARE (7) is anti-stabilizing, i.e., $\sigma(A_+) \subset \mathbb{C}^+$. \square

We conclude from [Proposition 14](#) that for systems $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^w$ given in minimal DV-representation $\mathfrak{B}_{DV}(A, B, C, D)$ the main result of this paper, [Theorem 3](#) remains valid if we replace ‘strictly Σ -dissipative on \mathbb{R}_- ’ by ‘ Σ -dissipative on \mathbb{R}_- ’, provided that $D^T \Sigma D > 0$ and the Hamiltonian matrix H given by (19) has no imaginary axis eigenvalues.

Remark 15. We note that if the condition $D^T \Sigma D > 0$ does not hold or if the Hamiltonian H has purely imaginary eigenvalues, then the optimal future and past trajectories do no need to exist (by singularity of the underlying linear quadratic problems (3) and (4)). Hence, in this case the operators Γ_- and Γ_+ are not even well defined.

Remark 16. We now briefly compare the results in this paper with those obtained in Weiland [11]. Essentially, in [11] it is assumed that $\Sigma = I$, the $w \times w$ identity matrix. Since $\sigma_+(I) = w$, which in general is unequal to the input cardinality $\mathfrak{m}(\mathfrak{B})$, his setup is not a special case of ours (in order to capture the PR and BR cases, we assume in this paper that $\mathfrak{m}(\mathfrak{B}) = \sigma_+(\Sigma)$). In that sense, the work in [11] diverges from ours. The operator Γ_- from past to future behavior in [11] does coincide with ours. However, in [11] the operator $\Gamma_- \Gamma_-^*$ is trivially nonnegative because both the past and future behavior are genuine inner product spaces, while in our work this nonnegativity follows from Σ -dissipativity on \mathbb{R}_- together with the assumption $\mathfrak{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. In the end, Weiland in fact proves that the singular values of the map Γ_- (for $\Sigma = I$) coincide with the LQG-characteristic values, i.e. the square roots of the eigenvalues of $-K_-^{-1} K_+$, where K_- and K_+ are the minimal and maximal real symmetric solutions of the ‘control algebraic Riccati equation’ of LQG control. Thus, [11] gives an operator characterization of the LQG-characteristic values, whereas in our paper we give such a characterization for the PR and BR characteristic values.

Remark 17. In [12], yet another operator characterization of the PR and BR characteristic values was given. Given a PR or BR input–output system, it was first shown that the system admits a so-called Σ -normalized DV-representation. Then, the PR and BR characteristic values were characterized in terms of the singular values (again in an indefinite inner product sense) of the Hankel operator from driving variable to input–output pair of this DV-representation. For details we refer to [12], Corollary 5.2.

5. The special cases of positive real and bounded real balancing

We will now apply our previous results to the special cases of positive real (PR) and bounded real (BR) balancing. Consider the system given by the minimal representation

$$\dot{x} = Ax + Bu, \quad y = \bar{C}x + \bar{D}u. \quad (20)$$

1. *Positive real case.* Assume that this input–output system is strictly positive real. The associated PR characteristic values are the square roots of the eigenvalues of the matrix $P_+^{-1}P_-$, where P_+ and P_- are the maximal and minimal real symmetric solutions of the PR algebraic Riccati equation

$$A^\top P + PA + (PB - \bar{C}^\top)(\bar{D} + \bar{D}^\top)^{-1}(B^\top P - \bar{C}) = 0. \quad (21)$$

As noted before, a driving variable representation of the input–output behavior represented by (20) is given by $\mathfrak{B}_{DV}(A, B, C, D)$, with

$$C := \begin{pmatrix} 0 \\ \bar{C} \end{pmatrix}, \quad D := \begin{pmatrix} I \\ \bar{D} \end{pmatrix}.$$

The supply rate matrix Σ is in this case given by $\Sigma = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$. By substituting C, D and Σ in the ARE (7), we indeed obtain the PR algebraic Riccati equation (21). As an immediate consequence of Theorem 13 we therefore obtain that the PR characteristic values of the input–output system (20) coincide with the $\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$ -characteristic values of the input–output behavior associated with system (20).

By Proposition 14, this result remains valid if we relax strict positive realness to positive realness, provided however we make the additional assumptions that $\bar{D} + \bar{D}^\top > 0$, and the Hamiltonian H has no purely imaginary eigenvalues, where

$$H := \begin{pmatrix} A_1 & B_1 B_1^\top \\ C_1^\top \Sigma C_1 & -A_1^\top \end{pmatrix},$$

where $A_1 := A - B(\bar{D} + \bar{D}^\top)^{-1}\bar{C}$, $B_1 := B(\bar{D} + \bar{D}^\top)^{-\frac{1}{2}}$, $C_1 := \begin{pmatrix} 0 \\ \bar{C} \end{pmatrix} - \begin{pmatrix} I \\ \bar{D} \end{pmatrix}(\bar{D} + \bar{D}^\top)^{-1}\bar{C}$.

2. *Bounded real case.* Next we look at the bounded real case. Assume that the input–output system (20) is strictly bounded real. Then the associated BR characteristic values are the square roots of the eigenvalues of the matrix $P_+^{-1}P_-$, where P_+ and P_- are the maximal and minimal real symmetric solutions of the BR algebraic Riccati equation

$$A^\top P + PA + \bar{C}^\top \bar{C} + (PB + \bar{C}^\top \bar{D})(I - \bar{D}^\top \bar{D})^{-1} \times (B^\top P + \bar{D}^\top \bar{C}) = 0. \quad (22)$$

The supply rate matrix Σ is in this case given by $\Sigma = \begin{pmatrix} I_m & 0 \\ 0 & -I_p \end{pmatrix}$. Again, by substituting C, D and Σ in the ARE (7) we obtain the BR algebraic Riccati equation (22), so the BR characteristic values of the input–output system (20) coincide with the $\begin{pmatrix} I_m & 0 \\ 0 & -I_p \end{pmatrix}$ -characteristic values of the input–output behavior associated with system (20).

Again, by Proposition 14, this result remains valid if we relax strict bounded realness to bounded realness, provided however we make the additional assumptions $I - \bar{D}^\top \bar{D} > 0$ and the Hamiltonian H associated with the supply rate Σ has no purely imaginary eigenvalues. We leave the details to the reader.

Remark 18. At this point it is unclear how to extend these results to the positive real case if the assumptions on $\bar{D} + \bar{D}^\top$ and the Hamiltonian H are not satisfied. The same remark holds for the bounded real case. In these cases the operators Γ_- and Γ_+ are not defined and the definition of Σ -characteristic value (Definition 4) collapses. This issue is left for future research.

6. Conclusions

In this paper we have studied strictly half line dissipative behaviors whose input cardinality is equal to the positive signature of the supply rate. Two important special cases are the input–output behavior of strictly positive real systems and of bounded real systems. We have introduced a linear operator from past to future behavior, and have introduced the Σ -characteristic values of the given behavior in terms of this operator. These values can be interpreted as singular values, but we have noted that this should be interpreted in a generalized sense, since the co-domain of the operator is an indefinite inner product space. We have shown that the Σ -characteristic values can be computed in terms of the extremal solutions of the algebraic Riccati equation associated with a driving variable representation of the behavior. Using this, we have concluded that the PR and BR characteristic values appearing in PR and BR preserving balanced truncation coincide with the singular values of the linear operator that assigns to each past input–output trajectory the unique future continuation that extracts the maximal amount of supply from the system.

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